Lecture 4

Tuesday, 1 September 2026 6:21 PM

Recap: Fisher market for goods:

- each agent $i \in N$ has a valuation density fn. $f_i : C \to \mathbb{R}_+$
- there is a price density fn. $p : C \to \mathbb{R}_+$
- each agent has 1 Re to spend

Each agent chooses a piece $S_i \subseteq [0,1]$ so that $p(S_i) \leq 1$ & S_i maximizes $v_i(S_i)$

demand set of i : $\{ S \subseteq [0,1] : p(S) \leq 1,\ S$ maximizes $v_i(S)$ subject to this$\}$

Claim: S is in i's demand set iff $\exists \alpha_i$ s.t. upto measure 0 sets,

$\{x : v_i(x)/p(x) > \alpha_i\} \subseteq S$,

$\{x : v_i(x)/p(x) < \alpha_i\} \cap S = \phi$,

and either $\alpha_i = 0$, or $p(S) = 1$.

(prove yourself)

A Competitive Equilibrium from Equal Incomes (CEEI) is a price density fn. p & an allocation $A = (A_1, \dots, A_n)$ such that:

1. A partitions the cake
2. Each A_i is in i's demand set, given prices p
3. $p(A_i) = 1$

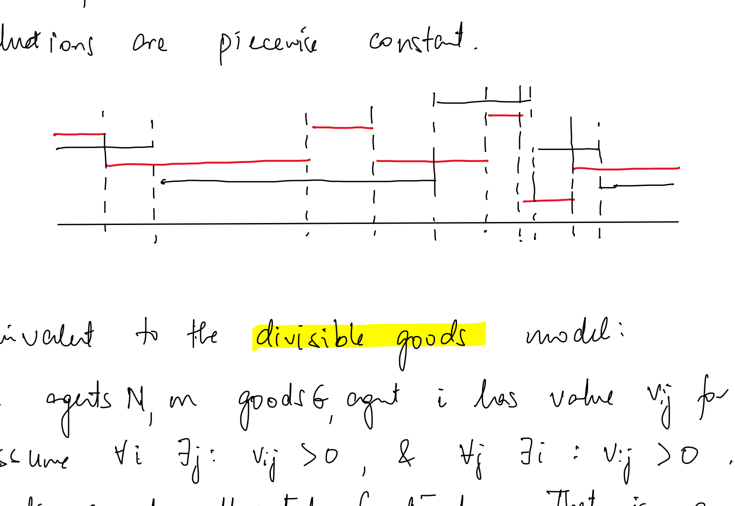
Lemma 1: A CEEI is EF.

Lemma 2: If agents are non-satiable (i.e., $v_i(x) > 0$ $\forall x \in [0,1]$) then a CEEI is PO.

Today: how to find a CEEI.

A simpler model:

Valuations are piecewise constant.



Equivalent to the divisible goods model:

n agents N, m goods G, agent i has value v_{ij} for good j.

Assume $\forall i\ \exists j : v_{ij} > 0$, & $\forall j\ \exists i : v_{ij} > 0$.

Goods can be allocated fractionally. That is, an allocation

$x = (x_1, \dots, x_n)$

where each x_i is a vector $x_{i1}, \dots, x_{im}$

& $x_{ij} \in [0,1]$ is the fraction of good j that agent i receives.

Further $\sum_{i \in N} x_{ij} = 1$ $\forall j \in G$ (each good is completely allocated)

(note that we don't need RW queries now, can talk directly about computational complexity).

An EF allocation is trivial now.

What about an EF + PO allocation?

A Welfare Objective

Consider the following welfare objectives: choose allocation $x = (x_1, \dots, x_n)$ to maximize

1. $\frac{1}{n} \sum_i v_i(x_i)$ (the utilitarian social welfare)
2. $\min_i v_i(x_i)$ (the egalitarian social welfare)
3. $\left(\prod_i v_i(x_i) \right)^{1/n}$ (the Nash welfare)

Of these, only the Nash welfare is scale invariant.

This problem: $\max \left(\prod_i v_i(x_i) \right)^{1/n}$ s.t. $\forall j, \sum_{i \in N} x_{ij} \leq 1$ & $\forall i,j,\ x_{ij} \geq 0$

is called the Eisenberg-Gale convex program.

Theorem: An allocation that maximizes the NW is a CEEI.

Conversely, any CEEI where every agent is non-satiated (i.e., $v_i(A_i) < 1$) maximizes NW.

Note that it is immediate that the NW maximizing allocation is PO.

Alternatively, we can write it as:

$$\max \frac{1}{n} \sum_i \log v_i(x_i)$$

$$\text{s.t. } \forall j \quad \sum_{i \in N} x_{ij} = 1 \qquad \times\ p_j$$

$$\forall i,j \quad x_{ij} \geq 0 \qquad \times\ \mu_{ij}$$

This is a convex program: linear constraints, concave maximization objective. Hence $\exists$ a maximum. Can be solved to arbitrary precision in polynomial time.

Claim: $\exists$ a unique maximum vector of optimal utilities $(v_1^*, \dots, v_n^*)$ (the allocation itself may not be unique)

(prove yourself)

MNW $\Rightarrow$ CEEI: will show $\exists$ prices $\{p_j\}_{j \in G}$ s.t. $\forall i$,

1. $\sum_{j \in G} p_j x_{ij} = 1$,
2. $x_i \in \arg\max_{z_i : \sum_j p_j z_{ij} \leq 1} \{ v_i(z_i) \}$

A brief primer on convex duality.

A convex program: $\min f(x)$ $\qquad$ f, g_j are convex.

$\forall j \in [m] \quad g_j(x) \leq 0 \quad \times \lambda_j$

(can be solved in polynomial time to arbitrary precision)

$$\mathcal{L}(x, \lambda) = f(x) + \sum_j \lambda_j g_j(x), \quad \lambda \geq 0$$

& note that for any feasible x (i.e., $g_j(x) \leq 0$ $\forall j$), $\lambda \geq 0$,

$$\mathcal{L}(x, \lambda) \leq f(x)$$

Now define the dual: $h(\lambda) = \inf_x \mathcal{L}(x, \lambda)$

Then $\boxed{h(\lambda) \leq \mathcal{L}(x, \lambda) \leq f(x)}$ WEAK DUALITY

Note so far we have not used convexity, to get weak duality.

KKT Conditions (Karush-Kuhn-Tucker):

x^* is optimal for the convex program iff $\exists$ multipliers / dual variables λ_j^* s.t.:

1. $\nabla_x \mathcal{L}(x, \lambda) = 0$ at x^*, λ^* (stationarity)
2. x^* is feasible ($g_j(x^*) \leq 0$ $\forall j$)
3. $\lambda^* \geq 0$
4. $\forall j,\ \mu_j g_j(x^*) = 0$ (complementary slackness)

Example: say $f(x) = x + (x-2)^2$, $\qquad x \in [0,1]$

$$-x \leq 0 \qquad \times\ l$$

$$x - 1 \leq 0 \qquad \times\ r$$

$$\mathcal{L}(x, l, r) = x + (x-2)^2 - lx + r(x-1)$$

$$\frac{\partial \mathcal{L}}{\partial x} = 1 + 2(x-2) - l + r = 0 \Rightarrow 2x = l - r + 3 \text{ or } x = \frac{1}{2}(l - r + 3)$$

$l, r \geq 0$, $x \in [0,1]$, $xl = r(x-1) = 0$

if $l > 0 \Rightarrow x = 0 \Rightarrow r = 0 \Rightarrow 0 = \frac{1}{2}(l+3)$ ×

if $r > 0 \Rightarrow x = 1 \Rightarrow l = 0 \Rightarrow 1 = \frac{1}{2}(3 - r)$, or $r = 1$ ✓

so by KKT condns., an optimal soln is $x = 1$, $l = 0$, $r = 1$.

Let's write down the Lagrangian:

$$\mathcal{L}(p, \mu, x) = \sum_i \log v_i(x_i) - p_j \left(\sum_{i \in N} x_{ij} - 1 \right) + \mu_{ij} x_{ij}, \quad p_j, \mu_{ij} \geq 0$$

By KKT conditions, x^* is optimal iff $\exists p^*, \mu^* \geq 0$ s.t.

1. x^* is feasible ($x^* \geq 0$, $\forall j \sum_{i \in N} x_{ij}^* \leq 1$)
2. $p^*, \mu^* \geq 0$
3. $\mu_{ij}^* x_{ij}^* = 0$ $\forall j$, $\quad p_j^* \left(\sum_{i \in N} x_{ij}^* - 1 \right) = 0$ $\forall j$
4. $\forall i,j,\ \left. \frac{\partial \mathcal{L}}{\partial x_{ij}} \right|_{x^*, p^*, \mu^*} = 0$

Adil, Dinesh
Pain is less
All ok
Meropenem

Lemma: Let x^* be an optimal soln. to the E-G convex program, & let p^*, μ^* be the KKT dual variables.

Then
1. $\forall i$, $\sum_j p_j^* x_{ij}^* = 1$
2. x_i^* maximizes $v_i(x_i^*)$, subject to 1

Proof:

$$\left. \frac{\partial \mathcal{L}}{\partial x_{ij}} \right|_{x^*, p^*, \mu^*} = 0 \Rightarrow \frac{v_{ij}}{v_i(x_i^*)} - p_j^* + \mu_{ij}^* = 0$$

Thus $\forall i, j$, $\quad p_j^* = \frac{v_{ij}}{v_i(x_i^*)} + \mu_{ij}^*$

$$\Rightarrow \sum_j p_j^* x_{ij}^* = \sum_j \frac{v_{ij} x_{ij}^*}{v_i(x_i^*)} + \sum_j \mu_{ij} x_{ij}^*$$

$$= 1 + \sum_j \mu_{ij} x_{ij}^*$$

... but $x_{ij}^* > 0 \Rightarrow \mu_{ij}^* = 0$

hence $\forall i$, $\sum_j p_j x_{ij}^* = 1$ — 1

Secondly, let $\alpha_i^* := v_i(x_i^*)$. Then

$$\forall i,j \quad \frac{v_{ij}}{\alpha_i^*} - p_j^* + \mu_{ij}^* = 0$$

or $\forall i, j$, $\frac{v_{ij}}{\alpha_i^*} \leq p_j^*$ or $\alpha_i \geq \frac{v_{ij}}{p_j}$

Suppose $\exists j : p_j^* = 0$. Then by stationarity,

$$\forall i, \quad \frac{v_{ij}}{v_i(x_i^*)} - p_j^* + \mu_{ij}^* = 0$$

or $\quad \frac{v_{ij}}{v_i(x_i^*)} = -\mu_{ij}^* \leq 0$

then $\forall i$, $v_{ij} = 0$, which contradicts our assumption.

Thus, $\alpha_i^* \geq \frac{v_{ij}}{p_j^*}$ $\forall i, j$

Further,

if $x_{ij}^* > 0$, $\quad \mu_{ij}^* = 0$

$$\Rightarrow \frac{v_{ij}}{\alpha_i^*} = p_j^* \text{ or } \alpha_i^* = \frac{v_{ij}}{p_j^*}$$

Thus, x_i^* maximizes $v_i(x_i^*)$, subject to $\sum_j p_j x_{ij}^* \leq 1$ ∎

Claim: There is a rational soln. to the E-G convex program (w/o proof)

Q. Can it be that for some $j \in G$, $\sum_i x_{ij} < 1$? what are prices then? α_i's?

Q. Given optimal x^*, compute prices?